Renormalized perturbation Theory: a missing chapter

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International Journal of Geometric Methods in Modern Physics

RENORMALIZED PERTURBATION THEORY: A MISSING CHAPTER

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Received (Day Month Year)
Revised (Day Month Year)

Renormalized perturbation theory à la BPHZ can be founded on causality as analyzed by H. Epstein and V. Glaser in the seventies.

Here, we list and discuss a number of additional constraints of algebraic character some of which have to be considered as parts of the core of the BPHZ framework.

Keywords: quantum field theory; renormalized perturbation theory.

1. Introduction

Why a talk on Renormalized Perturbation Theory (RPT) in 2008? The consensus established in the 70's under the acronym BPHZ[a] is part of elementary particle physicists' theoretical equipment.

Yet, the corresponding literature is hard to penetrate for a mind endowned with good logical connections - typically that of a professional mathematician. This state of affairs may be assigned, in parts, to some fuzziness about the connection between the operator version and the functional version (à la Feynman) of quantum mechanics, in this context.

In standard textbooks[5], the latter are both usually described in formal terms which are most of the time not subject to any mathematical formalization [b].

Parallel to the establishment of BPHZ, and following the path indicated by E.C.G. Stueckelberg[6][7], N.N. Bogoliubov[1] and coworkers, H. Epstein and

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[a]BPHZ: Bogoliubov, Parasiuk, Hepp, Zimmermann, see [1], [2], [3], [4].

[b]e.g. equal time commutation relations for interacting fields, writing down sharp-time time ordered products.

V. Glaser [8] have opened the road -also in the 70's- to such a mathematization. This path has been scarcely followed for the main following two reasons, as it seems:

- BPHZ has been found tight enough within the physics community mostly concerned with a large variety of interesting topics.
- E.G.'s constructions have remained beyond a fence which keeps their work isolated form concrete application of BPHZ, which rely on the algebraic structure of RPT mostly studied by Z and coworkers[3][9][10][11].

The potentialities of EG have however been tested on classes of popular models -mostly gauge theories, abelian and non abelian- by G. Scharf (Univ. fo Zürich) and coworkers[12][13].

¿From the philosophical point of view, EG as well as Z (and K. Symanzik 1970) formulate RPT in a way that does not require the use -and removal- of regularizations, in much the same spirit as was adopted, in concrete cases, to derive properly substracted dispersion relations[14].

One of the heroic founders of RPT, J. Schwinger[15], was obviously attracted by such an approach which he baptized "source theory" -without realizing that EG had at least settled the question to all orders of RPT -as a means of curing the trauma caused by (ultra-violet UV) infinities.

To try and cut a long story short, this program has diffused away from Zürich (where G. Scharf has retired) to Hamburg and Göttingen, under the joined leadership of Michael Dütsch (abandoned by Zürich) and Klaus Fredenhagen, coworkers and students[16][17][18][19][20][21].

The road between EG and BPHZ has proved longer than expected.

I will try to summarize some of what has been achieved for RPT on Minkowski space (which if a very small part of the whole).

2. Free fields

It is customary to start from fields whose equations of motion derive from a Lagrangian.

We will refrain from doing so a priori, because of the famous example of the free Maxwell field, linear in the creation and annihilation operators for photons with two helicity states, as derived form Wigner's representation of the Poincaré group for zero mass, helicity ± 1.

The free Maxwell equations

$$\partial_\mu F_{\mu\nu} = \partial_\mu \tilde{F}_{\mu\nu} = 0 (\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}) \tag{1}$$

do not derive form a Lagrangian.

For algebraic reasons which will become manifest as we go along we shall however land very close to this restricted class of free fields.

For what concerns us, these are Wightman fields[22][23][24]. We shall however give up the assumption that the Fock space under consideration has a positive definite Hilbert space metric -allowing for fields which have proven useful in the framework of gauge theories-. The connection between spin and statistics can then be jeopardized.

Given a finite set of free fields $\hat{\varphi}$, the (Z_2 graded-) commutative algebra $\hat{W}$ of local Wick polynomials of the field and their derivatives offers a quantum analog for the space of local interactions[23].

"As is well known"[20],

$$\hat{\mathcal{W}} \sim \mathcal{P}/J(\mathcal{E}) \tag{2}$$

where $\mathcal{P}$ is the algebra of local polynomials of similarly labelled classical fields and $J(\mathcal{E})$ the ideal generated by the equations of motion fulfilled by $\hat{\varphi}$.

This "well known" fact has however to be taken with a grain of salt because, if $\hat{\varphi}$ is to take values in a representation space of $SL2\mathcal{C} \times SL2\mathcal{C}$- with regard to Lorentz covariance- or some "internal" compact global symmetry group G, $\mathcal{P}$ itself comes as a quotient of $\mathcal{P}^{\bigoplus}$ (free algebra generated by monomials $\equiv$ [EG]'s supermultiquadriindices) by an ideal of relations fulfilled by monomials

$$\mathcal{J}(SL2\mathcal{C} \times SL2\mathcal{C}) \quad (resp\ \mathcal{J}(G)). \tag{3}$$

For instance $\mathcal{J}(SL2\mathcal{C} \times SL2\mathcal{C})$ is generated by the relations which express the linear dependence of three vectors in 2-dimensional space[25][26][27][28]

$$0 = \upsilon_1 \wedge \upsilon_2 \wedge \upsilon_3 = \upsilon_1(\upsilon_2\upsilon_3) + \upsilon_2(\upsilon_3\upsilon_1) + \upsilon_3(\upsilon_1\upsilon_2) \tag{4}$$

where $(\upsilon_i\upsilon_j) = \upsilon_i^\alpha \;\; \varepsilon_{\alpha\beta} \;\; \upsilon_j^\beta, \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha},\ \varepsilon_{12} = +1.$

This caveat will take its strength from linearity, resp. multilinearity, requirements we shall be inclined to enforce on the following constructions.

Regarding the quotient by $\mathcal{J}(\mathcal{E})(resp\ \mathcal{J}^{\bigoplus}(\mathcal{E}))$, we shall only consider the simplest situation where

$$\begin{gathered}\mathcal{P} = [\mathcal{W}] \bigoplus \mathcal{J}(\mathcal{E}) \\ resp\ \ \mathcal{P}^{\bigoplus} = [\mathcal{W}]^{\bigoplus} \bigoplus \mathcal{J}^{\bigoplus}(\mathcal{E})\end{gathered} \tag{5}$$

which is almost as strong as requiring that the fields derive from a non degenerate Lagrangian. Fields fulfilling such a strong requirement will be baptized regular fields.

Fields which are solutions of a hyperbolic system for which there are unconstrained Cauchy data on fixed time hypersurfaces are regular.

This is as broad as we could find a substitute for the usual canonical formalism.

(N.B.: Restriction at fixed time is legal for distribution solution of a hyperbolic equation).

In view of these delicacies we have to leave the operator framework for a functional set up where[18]

$\hat{\varphi} \rightarrow \varphi$ similarly labelled classical field

$$
\begin{aligned}
:\hat{m}^{\alpha}(\hat{\varphi}, D\hat{\varphi}): &\to m^{\alpha}(\varphi, D\varphi) \quad \text{classical monomial} \\
::\hat{m}^{\alpha_1}:(\hat{\varphi}, D\hat{\varphi})(x_1)\ldots :\hat{m}^{\alpha_n}:(\varphi, D\varphi)(x_n): &\to m^{\alpha_1}(\varphi, D\varphi)(x_1)\ldots m^{\alpha_n}(\varphi, D\varphi)(x_n) \\
\text{operator product} &\quad \text{ordinary product} \\
:\hat{F}(\tilde{\varphi})::\tilde{G}(\tilde{\varphi}): &\to F(\varphi) * G(\varphi) \\
&= F(\varphi) exp\left[i\hbar \int dxdy \frac{\overleftarrow{\delta}}{\delta\varphi(x)} \Delta^{+}(x-y) \frac{\overrightarrow{\delta}}{\delta\varphi}(y)\right] G(\varphi) \\
(\Omega, \hat{\varphi}(x)\hat{\varphi}(y)\Omega) &= i\hbar\Delta^{+}(x-y) \\
(\Omega, \hat{F}(\hat{\varphi})\Omega) &\Rightarrow \langle F(\varphi)\rangle = F(\varphi)\mid_{\varphi=0} \\
\Omega : \text{vacuum state in Fock space}
\end{aligned} \tag{6}
$$

φ will be taken among smooth functions. Its growth properties become important in the discussion of the so called adiabatic limit which will not be touched upon here.

F will be taken form the space of functionals of φ.

Functionals with arguments from $\mathcal{P}^{\oplus}$ read:

$$
F = \sum_n \int dx_1 \ldots dx_n \; F_{\alpha_1 \ldots \alpha_n}(x_1, \ldots x_n) m^{\alpha_1}(\varphi)(x_1) \ldots m^{\alpha_n}(\varphi)(x_n) \tag{7}
$$

where the $F_{[\alpha]}$'s are distribution kernels.

3. EG's causality condition

In view of the algebraic delicacies we have mentioned, we shall work within $\mathcal{P}^{\oplus}$ and functionals thereof. The effect of interactions is described by a scattering operator in Fock space. The corresponding functional will be constructed as a formal power series in a set of smooth coupling functions $\{g_\alpha\}$ associated with a monomial basis $\{m^\alpha(\varphi, D\varphi)\}$ of $\mathcal{P}^{\oplus}$.

Following tradition, we shall write

$$
\begin{aligned}
S(g,\varphi) &= \sum_{n=0}^{\infty} S_n(g,\varphi) \\
&= 1 + \frac{i}{\hbar} \int_{M_\tau} d^4x \sum_\alpha g_\alpha(x) m^\alpha(\varphi, D\varphi)(x) \\
&+ \sum_{h\geqslant 2} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{M_4^{\times n}} dx_1 \ldots dx_n g_{\alpha_1}(x_1) \ldots g_{\alpha_n}(x_n) \\
&\cdot \; T\left(m^{\alpha_i}(\varphi, D\varphi)(x_1) \ldots m^{\alpha_n}(\varphi, D\varphi)(x_n)\right)
\end{aligned} \tag{1}
$$

the coefficients of which will be determined recursively.

By construction, they are symmetric in their arguments (which reflects the commutativity of space time M_4, here, Minkowski space).

The g's are chosen with compact support or in $\mathcal{S}$. This is partly a technical convenience, some of the physical content of the construction being concerned with the limit $g_\alpha(x) \to g_\alpha$ (cst) for some α's, (the so called adiabatic limit).

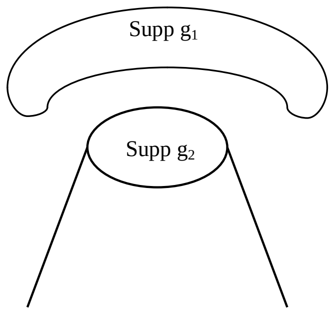
In order to conform with usage, we have kept $\hbar$ ("Planck's constant") as a formal variable to which are attached some combinatorial properties of the construction.

The causality requirement is

$$S(g_1 + g_2, \varphi) = S(g_1, \varphi) * S(g_2, \varphi) \tag{2}$$

for $suppg_1, \gtrsim suppg_2$

$\equiv suppg_1 \bigcap suppg_2 + \bar{V}_- = \emptyset)$ where $\bar{V}^-$ is the closed past light cone). This is called causal factorization.



This is turned by EG into the double recursion hypothesis:

$$\begin{array}{rcl} 1)\ T^{\alpha_{[n]}}(X_n) & = & T^{\alpha_{[I]}}(X_I) * T^{\alpha_{[I']}}(X_{I'}) \\ & X_I \gtrsim X_{I'} & \\ 2)\ \ [T^{\alpha_{[n]}}(X_n) * T^{\alpha_{[n']}}(Y_{n'})] & = & 0 \\ & X_n \sim Y_{n'} & \\ & n < N,\ n' < N & \end{array} \tag{3}$$

[Notation:

$$\begin{aligned} (1, \ldots n) &= [n] \\ X_n &= (x_1, \ldots x_n)\ ;\ x_i \in M_4 \\ \alpha_{[n]} &= (\alpha_1 \ldots \alpha_n) \end{aligned} \tag{4}$$

$$\begin{aligned} I \subset [n], \quad I' \subset [n], \quad I \cup I' = [n] \quad I \neq \emptyset \quad I' \neq \emptyset \quad I \cap I' = \emptyset \\ X_I \gtrsim X_{I'} \equiv \{x_i \gtrsim x_{i'}\ i \in I\ i' \in I'\} \\ X_n \sim Y_{n'} \equiv \{X_n \gtrsim Y_{n'} \text{ and } Y_{n'} \gtrsim X_n\}\,. \end{aligned} \tag{5}$$

At order N, one constructs

$$T_I^{\alpha_{[N]}}(X_N) = T^{\alpha_{[I]}}(X_I) * T^{\alpha_{[I']}}(X_{I'}),\ I \cup I' = [N] \tag{6}$$

whose existence is guaranteed by E.G.'s th0 (which states that Wick polynomials of free fields can be multiplied by translation invariant distributions).

Using (1) and (2), one proves

$$T_I^{\alpha[N]}(X_N) = T_J^{\alpha[N]}(X_N) \text{ in} C_I \cap C_J \tag{7}$$

where

$$C_I = \{X_I \gtrsim X_{I'}\} \,. \tag{8}$$

¿From geometry

$$\bigcup_I C_I = M_4^{\times N} \backslash D_N \tag{9}$$

where D_N is the diagonal $\{x_1 = \ldots = x_n\}$.

Using (K. Fredenhagen) a partition of unity $\{\alpha_I\}$ subordinated to the covering $\{C_I\}$ of $M_4^{\times N} \backslash D_N$, one defines there[29]

$$\widetilde{T}^{\alpha_{[N]}}(X_N) = \sum_I \alpha_I(X_N)\, T_I^{\alpha_{[N]}}(X_N) \tag{10}$$

which, by 1) and 2), is shown to be independent of the choice of $\{\alpha_I\}$.

Renormalization consists of extending $\widetilde{T}^{\alpha_{[N]}}(X_N)$ to all of $M_4^{\times N}$ [30][31].

This involves several steps which constitute the hard core of EG.

(i) reducing to scalars : one looks for solutions which fulfill the Wick Taylor expansion formula:

$$\begin{aligned} T\,(m^{\alpha_1}(\varphi, D\varphi)(x_1)\ \ldots\ m^{\alpha_n}(\varphi, D\varphi)(x_n)) = \\ \sum_{\beta\cup\gamma=\alpha} \quad \langle T\ m^{\beta_1}(\varphi, D\varphi)(x_1)\ldots m^{\beta_n}(\varphi, D\varphi)(x_n)\rangle \\ \times\, m^{\gamma_1}(\varphi, D\varphi)(x_1)\ldots m^{\gamma_n}(\varphi, D\varphi)(x_n) \end{aligned} \tag{11}$$

(ii) Reduce T to T^c (c for "connected" through a "log" algorithm)

(iii) Enforce translation invariance, by construction.

(iv) Show that $\langle T^c \rangle$ can be extended (one way is to use regularizations and renormalization).

(v) Classify the ambiguity of the extensions. They have support D_N. In the operator formalism there is a theorem ([8], [32]) which guarantees that it is of the form

$$\Delta T^{\alpha_{[N]}}(X) = \sum_\beta P_\beta^{\alpha[N]}(\partial)\delta(x_1 - x_N)\ldots\delta(x_{N-1}\ldots x_N) m^\beta(\varphi, D\varphi)(x_N) \tag{12}$$

where $P(\partial)$ is a differential operator with constant coefficients.

In the off shell formalism, one decides to restrict oneself to such ambiguities.

(vi) Power counting theory.

One can restrict the ambiguities so that

$$deg\ P^{\alpha[N]}_{\beta} \geq \left[\sum_{i=1}^{N} (\omega^{\alpha_i} - 4)\right] - (\omega^{\beta} - 4) \tag{13}$$

where the power counting index

$$\omega^{\alpha} = \omega(m^{\alpha}) = \omega\left(\Pi_i D^{\alpha_i} \varphi^i\right) = \sum_i \left[\omega(\varphi^i) + |\alpha_i|\right] \tag{14}$$

$\omega(\varphi^i)$ is computable form Δ^{+ij} $\omega^i + \omega^j - 4$ = naive scaling dimension of $\widetilde{\Delta}^{+ij}(p)$.

Remark: (Dütsch Fredenhagen[16]) if one imposes the Wick Taylor expansion formula to hold then the ambiguity ΔT^{α_N} has the above form.

This summarizes a very small (although already quite sizeable!) part of EG.

4. Further constraints

We have already seen one constraint one may wish to impose on T products (besides symmetry, translation covariance). One needs more before the space of ambiguities reaches a manageable size, but one has to be aware of the fact that the constraints one may wish to impose have to be shown compatible.

Among those which seem to be part of the game and have not found so far any replacement, are the following.

4.1. *Multilinearity (K. Fredenhagen)*[c]

The T's are multilinear in their arguments (e.g., relating $T2m^{\alpha}\ldots$ with $T(m^{\alpha}\ldots)$. This seems to be an "obvious" requirement to make, but it is absolutely not innocent. It is in particular this requirement which has lead us to go off shell (and even to $\mathcal{P}\oplus$).

Going back to the operator formalism in Fock space (Dütsch Boas)[20] then requires showing that one can construct T products in the functional formalism which belong to $\mathcal{F}\mathcal{J}(\mathcal{E})$ the $*$ ideal in the $*$ algebra of functionals) whenever one argument belongs to $\mathcal{J}(\mathcal{E})$.

One can do this in the particular case where one can write

$$\mathcal{P}^{\oplus} = \mathcal{J}^{\oplus}(\mathcal{E}) \oplus [\mathcal{W}] \tag{1}$$

[c]K. Fredenhagen. Many of the notions used here, besides multilinearity, are due to him, in writing or otherwise:

- identifying the product of functionals as a $*$ product,
- identifying AWI as a sufficient condition for the main theorem of renormalization to produce an unambiguous answer,
- identifying the Wick Taylor expansion formula as the solution of a Ward identity.

for some representative $[W]$ of W, in particular, in the case of "regular" fields (introduced for this purpose).

On the other hand we have at the moment nothing to say about the quotient by $\mathcal{J}(SL2\mathcal{C} \times SL2\mathcal{C})$ when Lorentz covariance is required.

4.2. *The Action Ward Identity (AWI)*

$$\partial^x_\mu T(m^\alpha(\varphi D\varphi(x)\ldots) = T\partial^x_\mu m^\alpha(\varphi, D\varphi)(x)\ldots \tag{2}$$

This has several names within BPHZ : solving the routing problem (W. Zimmermann), energy momentum conservation at each vertex of a renormalized Feynman Graph), $S(g)$ only depends on $S^1(g)$ not on the Lagrangian density ...

It can be imposed in the functional formalism (Dütsch Fredenhagen[16]), much less so in the operator formalism, the problem there being to find what subset of AWI is compatible with the quotient by $\mathcal{J}(\mathcal{E})$.

A representative $\mathcal{P}^\oplus_{bal}$ of $\mathcal{P}^\oplus \setminus Pol_+(\partial)\mathcal{P}^\oplus$ (polynomials without term constants) can be found, e.g. by going to Fourier transform, for each monomial and perform the change of variables $(p_1 \ldots p_n \to p_1 + \ldots p_n, p_1, \ldots p_{n-1})$ for a monomial involving $n \neq$ fields and, for identical fields, express symmetric polynomials in $p_1 \ldots p_n$ in terms of the symmetric functions $(\sum p_i, \sum_{i\neq j} p_i \otimes p_j + p_j \otimes p_i, \ldots)$.

Then one can prove

$$\mathcal{P}^\oplus = \mathcal{P}_{bal} \oplus Pol_+(\partial)\mathcal{P}_{bal}. \tag{3}$$

Take then an arbitrary solution for T, restrict it to arguments from $\mathcal{P}_{bal}$, and define it on $Pol_+(\partial)\mathcal{P}_{bal}$ by using AWI. Check this is a solution, which fulfills AWI, by construction. This has a very desirable consequence (Dütsch Fredenhagen[16]): in order to pass from one solution $S^I(g,\varphi)$, to another $S^{II}(g,\varphi)$, one can recursively absorb the ambiguities by which they differ at each order, into counterterms: $S^1(g,\varphi) \to S^1(g,\varphi) + (\Delta^{I,II} S^1(g,\varphi)$. This operation is in general not unique (due to the possibility to perform partial integrations).

It does become unique if the T's are restricted by AWI.

Then the corresponding S's can be written as functionals of g_{bal} and φ (and does not depend on the choice of g_{bal}) and the ambiguity which allow one to go from S^I to S^{II} acquires a natural group structure (the Stueckelberg Peterman renormalization group. M. Dütsch, K. Fredenhagen[16])

$$S^I\left(G^{I\ II}_{bal}(g_{bal}), \varphi\right) = S^{II}\left(g_{bal}, \varphi\right) \tag{4}$$

where $G^{I\ II}$ is a formal power series, local in g_{bal}).

One has

$$\begin{gathered} G^{I\ III}_{bal} = G^{I\ II}_{bal} \circ G^{II\ III}_{bal} \\ G^{I\ II}_{bal} \circ G^{II\ I}(g_{bal}) = g_{bal}. \end{gathered} \tag{5}$$

Where $\circ$ is the composition of formal power series.

N.B.: Power counting restrictions are essential for this to make sense (cf. Bourbaki Alg. Ch IV [33]).

The recursively defined ambiguities necessary at each order to have S^I match with S^{II}, collected into one formal power series local in $g_{bal}, \Delta^{I\ II}_{g_{bal}} (= O(g^2_{bal}))$ provide a parametrization

$$G^{I\ II}_{bal}(g_{bal}) = g_{bal} + \Delta^{I\ II}_{bal} \circ G^{I\ II}_{bal}(g_{bal}). \tag{6}$$

This is N. Bogoliubov's recursion relation[1]. It is solved by W. Zimmermann's forest formula[3] (cf. FM. Boas[17])

N.B.: This is an equation of the type $\underline{y} = \underline{x} + f(\underline{y})$ which, since Lagrange and Laplace has prompted a vast amount of literature [e.g. M. Haiman, W. Schmitt, Jour. Combin. Th. **50**, 172-185 (1989)[34]. [Thanks to S. Lazzarini for this reference].

The particular case of RPT has been closely scrutinized by A. Connes, D. Kreimer)[35].

Actually AWI is not only sufficient but necessary if one wants the ambiguities to be endowned with a group structure.

This is however not yet the renormalization group of BPHZ for which one has to reduce $\mathcal{P}^{(\oplus)}$ to $\mathcal{W}^{\oplus}$ and g_{bal} to g^{phys}_{bal} (the "physical" coupling constants to be defined).

This has been done in the case of regular fields.

4.3. *The Wick Taylor expansion formula*

already mentioned, optional.

4.4. *Connectedness and the $\hbar$ expansion* [d]

Connectedness has also has been used in the construction of a solution.

A combinatorial property of the Wick Taylor expansion formula is that the connected T^c's are formal power series in $\sqrt{\hbar}$ - and their vacuum expectation values formal power series in $\hbar$- at the heuristic unrenormalized Feynman graph level or at the level of the Wightman functions.

One may impose this as M. Dütsch, K. Fredenhagen, F. Brennecke do. Or, one may derive [38] it from a naturalness assumption according to which, within equivalence classes of free fields, isomorphisms should give rise to one to one correspondences between the corresponding T products. Here, apply this to φ and $\varphi\sqrt{\hbar}$, and $\varphi \to -\varphi$.

Combined with multilinearity, this puts constraints on ambiguities allowed by power counting.

[d]The combinatorics of the connected can be done either as in [36] or using Ruelle's $*$ product [37] which is the dual of the commutative coproduct standard for tensor algebras [33].

Naturalness also applies to Lorentz covariance and covariance under compact internal symmetry groups which may be parts of the attributes of the free fields.

5. General properties of R.P.T.

5.1. *Local insertions and the renormalized action principle*

Much of the combinatorial structure of RPT is connected with the renormalization group structure of the ambiguities.

The corresponding Lie algebra is the Lie algebra of "local insertions"

$$\Delta = \int d^g x \sum_\alpha \Delta_\alpha(g, Dg)(x) \frac{\delta}{\delta g_\alpha(x)} \tag{1}$$

(where g means g_{bal}), where the Δ_α's are local and constrained by power counting.

They are compatible with the causal factorization property

$$(\Delta S)(g_1 + g_2, \varphi) = (\Delta S)(g_1, \varphi) * S(g_2, \varphi) + S(g_1, \varphi) * (\Delta S)(g_2, \varphi) \tag{2}$$

$suppg_1 \gtrsim suppg_2$, as a consequence of the locality of Δ_α and the commutativity of Δ with the $*$ product.

As a result, $S + \Delta S$ fulfills causal factorization up to $O(\Delta^2)$.

Any Δ with this property therefore provides a ΔS which is an infinitesimal ambiguity, and therefore has the above form.

The totality of such Δ's (local derivations of the $*$ algebra of functionals) is not known. Some particular cases give rise to the so called "renormalized action principles" (Lowenstein[10], Lam[11], Breitenlohner, Maison[9], Dütsch[19]). For instance:

$$\begin{aligned}
\Delta_\varphi &= \int d^4x\ \Delta^i(g, Dg)(x)\ \frac{\delta}{\delta \varphi^i(x)} \\
\Delta_{t\varphi} &= \int d^4x\ \Delta(g, Dg)(t\varphi)^i\ \frac{\delta}{\delta \varphi^i(x)} \quad \text{if } [t, \Delta^+] = 0 \\
\Delta_{\mathcal{E}\varphi} &= \int d^4x\ \Delta(g, Dg)(x)\ \mathcal{E}(\partial)\ \varphi(x)\ \frac{\delta}{\delta \varphi(x)} \quad \text{if } \mathcal{E}(\partial)\Delta^+ = 0
\end{aligned} \tag{3}$$

The RAP's are usually presented within the framework of the Lagrangian formalism, for functionals introduced in that framework, in the adiabatic limit which we have not touched upon. Here, we shall limit ourselves to what we think is a key step in this direction, namely isolating the role of the free field equations of motion.

Assume one can find $\mathcal{P}_{bal}$ such that

$$\begin{aligned}
\mathcal{P}_{bal} &= \mathcal{P}_{phys} \oplus \mathcal{P}_{\mathcal{J}(\mathcal{E})} \quad \text{(all elements} \in \mathcal{J}(\mathcal{E})) \\
g_{bal} &= (g_{phys}, g_{\mathcal{E}})
\end{aligned} \tag{4}$$

This can be done (with some efforts for regular ($\supset$ Lagrangian) fields. One can find $\sum_c(g_{phys}, g_{\mathcal{E}}, \varphi)$ such that

$$\frac{\delta \Sigma_c}{\delta g_{\mathcal{E}}} \subset \mathcal{F}\mathcal{J}(\mathcal{E}) \tag{5}$$

the ideal generated by $\mathcal{J}(\mathcal{E})$ in the space of functionals.

This can be proved by recursion. Then, one has

$$S^c\left(g_{phys}, g_{\mathcal{E}}, \varphi\right) = \Sigma_c\left(G_{phys}(g_{phys}, g_{\mathcal{E}})\ G_{\mathcal{E}}(g_{phys}, g_{\mathcal{E}}), \varphi\right) \tag{6}$$

Differentiating with respect to $g_{\mathcal{E}}$ and taking vacuum expectation values, which annihilates terms belonging to $\mathcal{F}\mathcal{J}(\mathcal{E})$, one gets

$$\left\langle \frac{\delta S^c}{\delta g_{\mathcal{E}}} \right\rangle = \frac{\delta G_{phys}}{\delta g_{\mathcal{E}}} \left(\frac{\delta G_{phys}}{\delta g_{phys}}\right)^{-1} \left\langle \frac{\delta S_c}{\delta g_{phys}} \right\rangle \tag{7}$$

In other words, equations of motion multiplied by composite operators are expressible in terms of physical couplings. This contains RAP for the usual $\mathcal{Z}^c(g, J)$. (see section 5.3).

5.2. *Models*

Models are defined by submanifolds in the space of coupling functions which are stable under the action of the renormalization group.

This is an old idea (e.g., scalar electrodynamics has one more parameter than spinor QED, namely a quartic self coupling of the scalar field). This has been revived under the name "reduction of coupling constants" by W. Zimmermann, R. Oehme, K. Sibold and followers[3].

Most models are defined by a system of Ward identities in involution -modulo the proof that they can be fulfilled "without anomalies"-, besides AWI. Ex. $g_\alpha = 0$ $\omega^\alpha > 4$ (renormalized models).

5.3. *Contact with the conventional functional formalism*

Separating g and j. the coupling function of the field itself one can show that, given a Δ_F,

$$S\left(g, j=0; \varphi + \Delta_F * j\right) \mathrm{e}^{\frac{i}{\hbar}\langle j,\varphi\rangle + \frac{i}{2\hbar}\langle j, \Delta_F * j\rangle} \tag{8}$$

solves the causality program.

If the field derives from a Lagrangian $\left(\Delta_F^{*-1} = Pol(\partial)\delta = K\delta\right)$, the $*$ product is expressible in terms of j. For $\varphi = 0$

$$Z(g, j) = S\left(g, j=0, \Delta_F * j\right)\ \mathrm{e}^{\frac{i}{2\hbar}\langle j, \Delta_F^* j\rangle} \tag{9}$$

fulfills causality with respect to g, j with

$$* = exp\frac{i}{\hbar}\int dxdy \frac{\delta}{\delta \overleftarrow{j}(x)} \overleftarrow{K}_x \Delta^+(x-y) \overrightarrow{K}_y \frac{\delta}{\delta \overrightarrow{j}(y)}. \tag{10}$$

One then defines the IPI generator $\Gamma(g, \varphi)$ by Legendre transform.

This can be done starting from $S^c(g, o; \varphi)$ itself without the need of a Lagangian:

$$S^c(g, o; \varphi) = \Gamma'(g, \phi) - \frac{1}{2}\left(\frac{\delta\Gamma'}{\delta\phi'}, \Delta_F * \frac{\delta\Gamma'}{\delta\phi}\right)|_{\phi = \varphi + \Delta_F * \frac{\delta\Gamma'}{\delta\phi}} \tag{11}$$

which is inverted by

$$\Gamma'(g,\phi) = S^c(g,0;\varphi) + \frac{1}{2}\left(\frac{\delta S^c}{\delta\varphi}, \Delta_F * \frac{\delta S^c}{\delta\varphi}\right)|_{\varphi=\phi-\Delta_F*\frac{\delta S^c}{\delta\varphi}} \tag{12}$$

(Γ' is the interacting part of Γ, i.e. the "effective interaction").

It is customary, in view of the adiabatic limit to eliminate from both Z and Γ the term independent of j, resp. φ, and to define Γ, as well as Γ' in such a way that it starts with terms quadratic in ϕ. Z^c has a term linear in j with coefficient $F = \Delta_F * \frac{\delta S^c}{\delta\varphi}|_{\varphi=0}$ which can be absorbed in the Legendre transform formula: it suffices to change ϕ into $\phi + F$ in the stationarity condition in order to have a Γ' which starts quadratically in ϕ.

6. Conclusion and outlook

There are still many "details" to be filled in, and, if possible, simplified in comparison with the existing proofs. There are also some "terrae incognitae". Here are some, belonging to either species.

6.1. $\mathcal{J}(\mathcal{E})$

Besides regular fields which may be slightly more general than those deriving from a non degenerate Lagrangian, the only case which has been looked at is that of the Maxwell field $F_{\mu\nu}$, which provides some understanding of the collection of exotic fields used in the perturbative treatment of gauge theories. This has been started by Michel Dubois-Violette who found a geometrical characterization of the Faddeev Popov ghost (mostly unpublished because of difficulties with the tip of the light cone). This has been continued (R.S. in "Fifty years of Yang Mills" 2004) but is by no means complete.

6.2. *Extending distributions*

It may be worthwhile studying $\langle\widetilde{T}^\lambda(x)\rangle = \langle\widetilde{T}^(\lambda x)\rangle$ for $\lambda > 0$ and its Mellin transform (cf. M. Bergére and YMP Lam circa 1975). This may provide a substitute for the dimensional complex parameter ϵ, without the group theoretical drawbacks associated with

6.3. $\mathcal{J}(SL2\mathbb{C} \times SL2\mathbb{C})$

6.4. *Connexity and 1PI*

There is some nice combinatorics associated with connexity (F. Patras, M. Schocker 2005[39], [40]). It would be nice to have a direct algebraic proof for the connexity of the Ruelle Araki products (EGS 75)[36] and a streamlining of the proof of the corresponding spectral properties. Same for 1PI.

6.5. *The adiabatic limit*

If the distributions $\langle\widetilde{T}(X)\rangle, \langle T(X)\rangle$ are defined as temperate, (the coupling functions belonging to $\mathcal{S}$, it may be desirable to study $\langle\widetilde{T}^{\lambda}(X)\rangle$ for $\lambda \to \infty$ by lifting them to a suitable compactification of $M_4^{\times|X|}$ and view the adiabatic limit as an extension problem on this compactification, to the compactification set (manifold).

The recursion procedure together with a characterization of the ambiguities may lead to an infrared renormalization group.

6.6. *From the off shell (functional) framework to the on shell (operator) framework*

The connection between the two set ups, which plagues quantum field theory deserves more care.

Acknowledgments

I have drawn much inspiration form G. Barnich, M. Bauer, C. Becchi, J. Bros, C. Brouder, M. Dubois-Violette (!), M. Dütsch, H. Epstein, K. Fredenhagen, G. Girardi, M. Henneaux, T. Hurth, C. Imbimbo, S. Lazzarini, F. Patras, G. Scharf, Th. Schücker, K. Sibold, I. Todorov, F. Thuillier, whom I thank for wasting some of their time responding to often silly questions.

The points of view expressed here reflect the desire to try and put together some of the field theory I learnt from R. Haag, R. Jost, G. Källen, D. Kastler, H. Lehmann, B. Schroer, J. Schwinger, K. Symanzik, A.S. Wightman, W. Zimmermann.

Appendix A. More general perturbations

During the colloquium Manuel Asorey asked the tantalizing question: can one describe interactions among more general local fields, beyond free fields. This is not only a natural question: it has been faced within the study of integrable perturbations of conformal fields. A systematic renormalized perturbation theory does not exist, however. We shall sketch out what seem to be the hardest obstructions to such a construction.

So let $\widehat{\varphi}$ be some Wightman fields (characterized by Wightman functions $W(x_1 \dots x_n) = (\Omega, \widehat{\varphi}(x_1) \dots \widehat{\varphi}(x_n)|0)$.

We need a space of interactions; it is natural to choose the Borchers class of $\widehat{\varphi}$. If $\widehat{\varphi}$ is described by some renormalized perturbation theory, the Borchers class will be labelled by local polynomials in $\widehat{\varphi}$ just as that of some corresponding free fields. The model we have in mind is described in EG:

$$V(g,h) = S^{-1}(g)\ S(g+h) \tag{A.1}$$

fulfills causal factorization for all g's, with respect to h if $S(g)$ is perturbatively defined with the latter causal property. We assume that the adiabatic limit $g(x) \to g$

(cst) exists. Together with the corresponding field $\widehat{\varphi}$, we have the local monomials $\widehat{m}^{\alpha}(\widehat{\varphi})$.

As in the free field case causal factorization allows to describe the recursive construction (in powers of h) as an extension problem through the diagonal at the level of functionals.

The construction gets stuck, with the present technology at the level of the reduction to scalars:

The Wick Taylor formula can be generalized following a construction due to AS Wightman and J. Challifour (1966 unpublished) modulo a slight generalization to include all $\widehat{m}^{\alpha}$'s Wick products $\vdots \cdot \vdots$ can be defined by the Wick Taylor formula:

$$\widehat{m}^{\alpha_1}(x_1)\ldots\widehat{m}^{\alpha_n}(x_n) = \sum_{\beta_i\cup\gamma_i=\alpha_i} \big(\Omega, \widehat{m}^{\beta_1}(x_1)\ldots\widehat{m}^{\beta_n}(x_n)\Omega\big) \vdots\widehat{m}^{\gamma_1}(x_1)\ldots\widehat{m}^{\gamma_n}(x_n)\vdots \tag{A.2}$$

with the convention $\widehat{m}^{\phi} = 1$, and there follows Wick's theorem

$$\vdots\, \widehat{m}^{[\alpha]}(X) \,\vdots \cdot \vdots\, \widehat{m}^{[\beta]}(Y) = \sum C^{[\alpha][\beta];[\alpha'][\beta']}(X\cup Y)\ \widehat{m}^{[\alpha'']\cup[\beta'']}(X\cup Y)$$
$$[\alpha_j \cup [\alpha'']] = [\alpha]$$
$$[\beta_j \cup [\beta'']] = [\beta] \tag{A.3}$$

where the contraction symbol $C^{[\alpha][\beta];[\alpha'][\beta']}(X\cup Y)$ is given by

$$C^{[\alpha][\beta];[\alpha'][\beta']} = \sum_k \prod_\kappa \left\langle \widehat{m}^{[\alpha'_\kappa][\beta'_\kappa]}(X\cup Y)\right\rangle^T$$
$$\bigcup_\kappa [\alpha'_\kappa] = [\alpha']$$
$$\bigcup_\kappa [\beta'_\kappa] = [\beta']$$
$$\alpha'_n \cdot \beta'_\kappa \ \text{ not simultaneously empty} \tag{A.4}$$

$\langle\ \rangle^T$ refers to the truncated expectation values.

The difficulties with these Wick prodcuts $\vdots\cdot\vdots$ is that, in general

(1) they only fulfill local commutativity (not, necessarily, full commutativity as $\vdots\ \vdots$)
(2) they are insufficiently renormalized (only vacuum contributors to divergences are substracted out), so that they do not fulfill theorem 0 of EG (multiplicability by translation invariant distributions.

Note however that using

$$\widehat{m}^{\alpha}(x) = \vdots\, \widehat{m}^{\alpha}(x)\, \vdots \tag{A.5}$$

(assuming $\langle \widehat{m}^{\alpha}(x)\rangle = 0$).

We can apply Wick's theorem to deduce

$$\langle \widehat{m}^{\alpha_1}(x_1)\ldots\widehat{m}^{\alpha_n}(x_n)\rangle = C^{\alpha_1\cdots\alpha_n}(x_1\ldots x_n) \tag{A.6}$$

where $C^{\alpha_1\cdots\alpha_n}(x_1\ldots x_n)$ is given by a sum of products of truncated functions which can be associated with a graph involving not only oriented lines joining two different vertices but also oriented circles joining larger subsets of points from $(x_1\ldots x_n)$.

These expressions are well defined as distributions because of the spectral properties which imply that the W^T's are boundary values of functions holomorphic in tubes in the difference variables, which allows to multiply boundary values.

The problem is now to define the corresponding time ordered products recursively. In the free field case, as already mentioned, there are two crucial properties:

(1) the Wick algebra is commutative
(2) it admits translation invariant distribution coefficients (THO)

Concerning 1) one may extend the above combinatorics to T products, which looks somewhat circular since it involves $\vdots\, T(\cdot)\, \vdots$'s for which there is no natural definition. 2) is even more problematic. In case of emergency, one may try to renormalize the time ordered versions of the contraction symbols. The corresponding combinatorics has been studied (excluding renormalization) in a recent article which the authors kindly sent me upon return from this conference: C. Brouder, A. Frabetti, F. Patras: "Decomposition into one particle irreducible Green functions in many body physics" : arXiv: 08033747, v.1 [cond-mat-str-el], 26 Mar 2008.

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